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A Necessary and Sufficient Contractivity Condition for the Fractal Transform Operator

Mehran Ebrahimi

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Abstract This paper revisits the concept of fractal image coding and the contractivity conditions of the fractal transform operator. All such existing conditions are only sufficient. This paper formulates a necessary and sufficient condition for the contractivity of the fractal transform operator associated to a fractal code. Furthermore, analytical results on the convergence of the fractal image decoding will be derived.

Keywords Fractal image coding · Fractal imaging · Contraction maps · Contractivity condition · Fixed point theorem · Convergence condition

1 Introduction

In the late-1980's, M. Barnsley of Georgia Tech, with coworkers and students, showed that sets of contractive maps with associated probabilities, called *Iterated Function Systems (IFS)*, could be used not only to generate fractal sets and measures but also to approximate natural objects and images [2]. This gave birth to *fractal image compression*, which would become a hotbed of research activity over the next decade. Historically, most fractal image coding research focused on its compression capabilities, and many theoretical aspects of the research were dismissed. This paper focuses on various unanswered theoretical questions in fractal image coding (with the acknowledgment that it no longer furnishes a competitive method of image compression).

In Sect. 2, we will present the precise definition of the fractal transform operator T and shall state that under suitable conditions the operator T is contractive. Hence, using Banach's contraction mapping principle, T has a unique fixed point $\mathbf{u}^\star$ such that $T(\mathbf{u}^\star) = \mathbf{u}^\star$.

There are a few existing *sufficient* contractivity conditions for the fractal transform operator in the literature [4, 7–9], which are *only sufficient* and *not necessary*. S.K. Alexander has stated [1] that such contractivity conditions are never checked in practice. Y. Fisher has suggested [7, 8] that based on "computational experiments" the operator T is converging if the scaling coefficients α_i (introduced in the next section) are in the interval $[-\sqrt{2}, \sqrt{2}]$. Furthermore, M. Ghazel has stated [10, 11] that "In fact, for block-based fractal schemes, it is indeed very difficult to derive a tight necessary contractivity requirement on the IFS coefficients".

More recently, M. Ghazel et al. [12] have stated that "There is one complication, however, in that the contractivity of the fractal transform operator T is dependent upon the α scaling coefficients. There is no simple relationship between the $\mathcal{L}^2$ contractivity factor of T and the α coefficients because of the local nature of the parent-child mappings". *An aim of this paper is to reveal that indeed there exists a simple relationship between the contractivity factor of the fractal transform operator T and the scaling coefficients which has been unknown for many years.*

In Sect. 3, we will obtain a necessary and sufficient condition for the contractivity of the fractal transform operator on $\mathbb{R}^{N\times N}$ in the main theorem of this paper. It will be shown that a corollary of the main theorem is the same sufficient contractivity conditions for the fractal transform operator, reported by B. Forte et al. [9] for the continuous case.

M. Ebrahimi (✉)
Department of Medical Biophysics, University of Toronto,
Imaging Research, Sunnybrook Health Sciences Centre, Toronto, Ontario, Canada
e-mail: mehran.ebrahimi@sri.utoronto.ca

In Sect. 4, we will ask the question whether contractivity of T is a requirement for the convergence of the fractal decoding scheme. A counter-example will be presented to show that the fractal decoding scheme may converge even if T is not a contraction.

Furthermore, we state analytical conditions under which the fractal decoding iteration converges. In addition, we derive a theorem in this *non-contracting yet converging* case, corresponding to the Collage Theorem (4). Finally, we will prove that this result generalizes the Collage Theorem and will refer to the result as "The Generalized Collage Theorem".

2 Block-Based Fractal Image Coding

More details on fractal image coding can be found in many places [2, 3, 5–8, 12, 14]. In this section, we outline the most important features of fractal image coding. Fractal image coding seeks to approximate an image by a union of spatially-contracted and greyscale-modified copies of subblocks of itself.

2.1 Fractal Image Encoding

Let an image of interest be represented by an image function $\mathbf{u}(x, y)$, denoted by $\mathbf{u} \in \mathbb{R}^{N\times N}$. Briefly, the result of the coding procedure is a contractive mapping T, the so-called *fractal transform* operator. The fixed point $\mathbf{u}^\star$ of T provides an approximation to $\mathbf{u}$. In other words, $\mathbf{u} \cong \mathbf{u}^\star = T(\mathbf{u}^\star)$. We need to take a few steps to define the operator T and study its properties.

2.1.1 Partitioning

First, consider a partition of $\Upsilon = [1, \dots, N] \times [1, \dots, N]$ on which the image is defined into nonoverlapping subblocks $\{C_i\}$, $i \in \mathfrak{C}$, referred to as *range (or child) subblocks*, such that $\Upsilon = \bigcup_{i\in\mathfrak{C}} C_i$. Also, consider a new partition of Υ by typically larger subblocks $\{P_j\}$, $j \in \mathfrak{P}$, referred to as *domain (or parent) subblocks*, such that $\Upsilon = \bigcup_{j\in\mathfrak{P}} P_j$. In this paper, we assume that the elements $\{P_j\}$ do not mutually overlap, which is not always a requirement in many texts. For example, we may assume that the square blocks C_i and $P_{J(i)}$ are, respectively of size $K \times K$ and $sK \times sK$, where each of these blocks belongs to the corresponding square grid partitions of Υ, for positive integers K and s. In this specific case, we need to assume sK divides N to guarantee the existence of these grids.

2.1.2 Block and Parameter Search

For each $i \in \mathfrak{C}$ associated with each range subblock C_i, one searches for the index $J(i) \in \mathfrak{P}$ of a corresponding domain subblock $P_{J(i)}$ in a manner that $\mathbf{u}(C_i)$ is well approximated by a spatially-contracted and greyscale-modified copy of $\mathbf{u}(P_{J(i)})$, i.e., $\mathbf{u}(C_i) \cong \phi_i(\mathcal{D}(\mathbf{u}(P_{J(i)})))$. Here, $\phi_i : \mathbb{R} \to \mathbb{R}$ are *greyscale maps* that operate on pixel intensities and are usually assumed to be affine, i.e., $\phi_i(t) = \alpha_i t + \beta_i$. Also, $\mathcal{D}$ is the decimation operator of factor s defined as $\mathcal{D} : \mathbb{R}^{sK\times sK} \to \mathbb{R}^{K\times K}$, for any $\mathbf{x} \in \mathbb{R}^{sK\times sK}$

$$(\mathcal{D}\mathbf{x})(m, n) = \frac{1}{s^2} \sum_{1\le m',n'\le s} \mathbf{x}(s(m-1)+m', s(n-1)+n'),$$
$$\forall 1 \le m, n \le K.$$

Therefore, for each $i \in \mathfrak{C}$ one finds some $(J(i), \alpha_i, \beta_i) = (j, \alpha, \beta)$, $j \in \mathfrak{P}$, $(\alpha, \beta) \in \Pi$ that minimize

$$\left\| \mathbf{u}(C_i) - \left\{ \alpha\mathcal{D}\big(\mathbf{u}(P_j)\big) + \beta \right\} \right\|. \tag{1}$$

Here, $\|.\|$ denotes the Frobenius (or Euclidean) norm, and $\Pi \subset \mathbb{R}^2$ denotes the feasible (α, β) parameter space, which will be suitably restricted. Note that such a minimization problem may have non-unique solutions. The solution of the above minimization problem is performed by exhaustive searching over all $j \in \mathfrak{P}$ for each $i \in \mathcal{C}$. For a domain-range block pair P_j/C_i, the optimal value of the α and β parameters may depend on the parameter space Π. Typically this may be accomplished by means of least-squares or constrained least-squares method. If $\Pi = \mathbb{R}^2$, i.e., when no constraint is assumed on the parameters α and β and the pair of blocks $\mathbf{u}(C_i), \mathcal{D}(\mathbf{u}(P_j))$ are respectively represented by vectors $\underline{\mathbf{u}}_{c_i}$ and $\underline{\mathbf{u}}_{p_j}$ of the same size, it is easy to show that (e.g. see [11]) the minimizing parameters of the above expression for some fixed $i \in \mathfrak{C}$ and $j \in \mathfrak{P}$ is given by $\alpha_i = \frac{\mathrm{Cov}(\underline{\mathbf{u}}_{p_j}, \underline{\mathbf{u}}_{c_i})}{\mathrm{Var}(\underline{\mathbf{u}}_{p_j})}$, $\beta_i = E(\underline{\mathbf{u}}_{c_i}) - \alpha_i E(\underline{\mathbf{u}}_{p_j})$, in the case that $\mathrm{Var}(\underline{\mathbf{u}}_{p_j}) \ne 0$. Furthermore, in the case that $\mathrm{Var}(\underline{\mathbf{u}}_{p_j}) = 0$ and the elements of $\mathbf{u}_{p_j}$ are all non-negative, $\alpha_i = 0$, $\beta_i = E(\underline{\mathbf{u}}_{c_i})$ are some non-unique minimizing parameters.

2.1.3 Fractal Transform Operator

We can summarize that a *fractal code* of the image $\mathbf{u}$ approximated in the outlined fashion consists of $\{J(i), \alpha_i, \beta_i\}$, for all $i \in \mathfrak{C}$. To introduce the *fractal transform operator* using the obtained fractal code, it is required to define one more ingredient, the so-called *block mapping*. For each $i \in \mathfrak{C}$, define a *block mapping* w_i, from $P_{J(i)}$ to C_i such that $C_i = w_i(P_{J(i)})$. If C_i is a block of size $K \times K$, and $P_{J(i)}$ is of size $sK \times sK$ then w_i relates every $s \times s$ block of $P_{J(i)}$ to the corresponding 1×1 pixel in C_i. More precisely, $\forall i \in \mathfrak{C}$ and $1 \le p_x, p_y \le sK$, $w_i(P_{J(i)}(p_x, p_y)) = C_i(\lceil \frac{p_x}{s} \rceil, \lceil \frac{p_y}{s} \rceil)$, which is equivalent to $\forall i \in \mathfrak{C}$, $(x, y) \in \Upsilon$, and $1 \le c_x, c_y \le s$,

$$w_i(P_{J(i)}(s(x-1)+c_x, s(y-1)+c_y)) = C_i(x, y).$$

Such a w_i is not 1–1 and by notations $w_i^{-1}(C_i)$ and $w_i^{-1}(x,y)$ for some $(x,y)\in C_i$, we respectively mean the inverse images of C_i and (x,y) under w_i. The former would simply be $P_{J(i)}$ and the latter is an $s\times s$ block in the corresponding $P_{J(i)}$. Hence,

$$\mathbf{u}(C_i)\cong\phi_i\big(\mathcal{D}\big(\mathbf{u}(P_{J(i)})\big)\big)=\phi_i\big(\mathcal{D}\big(\mathbf{u}(w_i^{-1}(C_i))\big)\big). \tag{2}$$

Let us now assume that we have computed a fractal code of an image function $\mathbf{u}$ according to (1). Because of the nonoverlapping nature of the partition by the range subblocks C_i's, we may define T,

$$\mathbf{u}(x,y)\cong(T(\mathbf{u}))(x,y)=\sum_{i\in\mathfrak{C}}\phi_i\big(\mathcal{D}\big(\mathbf{u}(w_i^{-1}(x,y))\big)\big), \tag{3}$$

for every $(x,y)\in\Upsilon$. The image function $\mathbf{u}$ is thus approximated as a sum of spatially-contracted and greyscale-distorted (ϕ_i) copies of its blocks. This T will be referred to as the *fractal transform operator* of $\mathbf{u}$. Under suitable conditions outlined (and further extended by the author) in the next sections, the operator T is contractive in $\mathbb{R}^{N\times N}$ [4]. As such, using Banach's contraction mapping principle, there will exist a unique fixed point $\mathbf{u}^\star\in\mathbb{R}^{N\times N}$ such that $T(\mathbf{u}^\star)=\mathbf{u}^\star$.

2.1.4 Collage Theorem

If the above approximation is a "good one", then the so-called *collage distance* $\|\mathbf{u}-T(\mathbf{u})\|$ is small. From the so-called "Collage Theorem" [4],

$$\|\mathbf{u}-\mathbf{u}^\star\|\le\frac{1}{1-con_{(T)}}\|\mathbf{u}-T(\mathbf{u})\|, \tag{4}$$

it then follows that if $\mathbf{u}$ is "close" to $T(\mathbf{u})$, then $\mathbf{u}$ is also close to $\mathbf{u}^\star$, the fixed point of T. Here, $con_{(T)}\in[0,1)$ denotes the contraction factor of T. The quantity $\|\mathbf{u}-\mathbf{u}^\star\|$ is the error of approximation of $\mathbf{u}$ by $\mathbf{u}^\star$.

2.2 Fractal Image Decoding

Once we have a fractal transform T, we may generate its fixed point $\mathbf{u}^\star$ by a simple iteration. Starting with an arbitrary image $\mathbf{u}_0$, one forms the iterations $\mathbf{u}_{n+1}=T(\mathbf{u}_n)$. In this decoding procedure, the image subblocks $\mathbf{u}_n(C_i)$ are replaced by modified copies $\phi_i(\mathcal{D}(\mathbf{u}_n(P_{J(i)})))$ according to (2). This yields,

$$\mathbf{u}_{n+1}(C_i)=\big(T(\mathbf{u}_n)\big)(C_i)=\alpha_i\mathcal{D}\big(\mathbf{u}_n(P_{J(i)})\big)+\beta_i,$$

starting with an arbitrary image $\mathbf{u}_0$. Banach's contraction mapping theorem guarantees that the sequence of images $\mathbf{u}_n$ converges to $\mathbf{u}^\star$, if T is a contraction.

3 Extending the Contractivity Conditions for the Fractal Transform Operator

In this section, we present a theorem that provides a necessary and sufficient condition for the contractivity of the fractal transform operator. Furthermore, we verify that the existing sufficient contractivity condition of the fractal transform operator, reported in [9] for the continuous case, can be derived as a corollary of our theorem in $\mathbb{R}^{N\times N}$. We state some important assumptions and prove a lemma before stating our main theorem.

Assumptions Throughout, we assume that the fractal transform T is defined according to (3) with the same variables, and the size of range and domain subblocks are respectively $K\times K$ and $sK\times sK$. We also employ the space $\mathbb{R}^{N\times N}$ with the Frobenius norm throughout.

Lemma 1 *For any* $\mathbf{y}\in\mathbb{R}^{N\times N}$ *and* $i\in\mathfrak{C}$, $\|\mathcal{D}(\mathbf{y}(P_{J(i)}))\|\le\frac{1}{s}\|\mathbf{y}(P_{J(i)})\|$.

Proof Assume that $P_{J(i)}$ is an $sK\times sK$ block and $\mathbf{y}(P_{J(i)})$ is represented by K^2 blocks each of size $s\times s$ by $y_{p,q}^{(m)}$ for $1\le p,q\le s$, and $1\le m\le K^2$. By this representation,

$$\|\mathcal{D}(\mathbf{y}(P_{J(i)}))\|^2=\sum_{m=1}^{K^2}\Bigg[\frac{1}{s^2}\sum_{1\le p,q\le s}y_{p,q}^{(m)}\Bigg]^2.$$

For every $1\le m\le K^2$, taking $a_{p,q}=y_{p,q}^{(m)}$ and $b_{p,q}=\frac{1}{s^2}$ in the Cauchy-Schwarz inequality

$$\Bigg(\sum_{1\le p,q\le s^2}a_{p,q}b_{p,q}\Bigg)^2\le\Bigg(\sum_{1\le p,q\le s^2}a_{p,q}^2\Bigg)\Bigg(\sum_{1\le p,q\le s^2}b_{p,q}^2\Bigg),$$

leads to

$$\Bigg[\frac{1}{s^2}\sum_{1\le p,q\le s}y_{p,q}^{(m)}\Bigg]^2\le\Bigg(\sum_{1\le p,q\le s^2}{y_{p,q}^{(m)}}^2\Bigg)\Bigg(\sum_{1\le p,q\le s^2}\frac{1}{s^4}\Bigg)$$
$$=\Bigg(\sum_{1\le p,q\le s^2}{y_{p,q}^{(m)}}^2\Bigg)\Bigg(\frac{1}{s^2}\Bigg).$$

Hence, taking the sum over m yields

$$\sum_{m=1}^{K^2}\Bigg[\frac{1}{s^2}\sum_{1\le p,q\le s}y_{p,q}^{(m)}\Bigg]^2\le\frac{1}{s^2}\sum_{m=1}^{K^2}\Bigg(\sum_{1\le p,q\le s^2}{y_{p,q}^{(m)}}^2\Bigg).$$

Substituting the equivalent values for the left and right sides gives,

$$\|\mathcal{D}(\mathbf{y}(P_{J(i)}))\|^2\le\frac{1}{s^2}\|\mathbf{y}(P_{J(i)})\|^2.$$

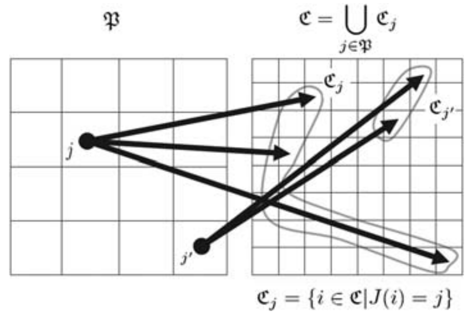

Fig. 1 Partitioning of $\mathfrak{C}$. Two elements of the partition are shown on the *right*

Finally taking the square root of both sides of this expression completes the proof. □

Now, we are ready to state the main theorem of this paper.

Theorem 1 (The Main Theorem) *A necessary and sufficient condition for the contractivity of* T *on* $\mathbb{R}^{N\times N}$ *is that* $\max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\} < 1$, *in which case* $con_{(T)} = \max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\}$, *where* $\mathfrak{C}_j = \{i \in \mathfrak{C} | J(i) = j\}$ *for any* $j \in \mathfrak{P}$.

Proof First we prove the sufficiency. For any $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^{N\times N}$,

$$\begin{aligned}
&\|T(\mathbf{u}) - T(\mathbf{v})\|^2 \\
&\quad= \sum_{i\in\mathfrak{C}}\sum_{x,y\in C_i}\big|\phi_i\big(\mathcal{D}\big(\mathbf{u}(w_i^{-1}(x,y))\big)\big) \\
&\qquad - \phi_i\big(\mathcal{D}\big(\mathbf{v}(w_i^{-1}(x,y))\big)\big)\big|^2 \\
&\quad= \sum_{i\in\mathfrak{C}}\big\|\phi_i\big(\mathcal{D}\big(\mathbf{u}(w_i^{-1}(C_i))\big)\big) - \phi_i\big(\mathcal{D}\big(\mathbf{v}(w_i^{-1}(C_i))\big)\big)\big\|^2.
\end{aligned}$$

Assuming $\phi_i(t) = \alpha_i(t) + \beta_i$ as before yields

$$\begin{aligned}
&\|T(\mathbf{u}) - T(\mathbf{v})\|^2 \\
&\quad= \sum_{i\in\mathfrak{C}}\alpha_i{}^2\big\|\mathcal{D}\big(\mathbf{u}(w_i^{-1}(C_i))\big) - \mathcal{D}\big(\mathbf{v}(w_i^{-1}(C_i))\big)\big\|^2 \\
&\quad= \sum_{i\in\mathfrak{C}}\alpha_i{}^2\big\|\mathcal{D}\big(\mathbf{u}(P_{J(i)})\big) - \mathcal{D}\big(\mathbf{v}(P_{J(i)})\big)\big\|^2 \\
&\quad= \sum_{i\in\mathfrak{C}}\alpha_i{}^2\big\|\mathcal{D}\big((\mathbf{u}-\mathbf{v})(P_{J(i)})\big)\big\|^2. \qquad (5)
\end{aligned}$$

Hence, by Lemma 1 taking $\mathbf{y} = \mathbf{u} - \mathbf{v}$, $\|T(\mathbf{u}) - T(\mathbf{v})\|^2 \le \sum_{i\in\mathfrak{C}}\frac{\alpha_i^2}{s^2}\|(\mathbf{u}-\mathbf{v})(P_{J(i)})\|^2$. Note that $\{\mathfrak{C}_j\}_{j\in\mathfrak{P}}$ forms a partition for $\mathfrak{C}$, i.e., $\mathfrak{C}_{j_1}\cap\mathfrak{C}_{j_2} = \emptyset$ for distinct $j_1, j_2 \in \mathfrak{P}$ and $\mathfrak{C} = \bigcup_{j\in\mathfrak{P}}\mathfrak{C}_j$ (see Fig. 1). Hence,

$$\begin{aligned}
\|T(\mathbf{u}) - T(\mathbf{v})\|^2 &\le \sum_{j\in\mathfrak{P}}\sum_{i\in\mathfrak{C}|J(i)=j}\left(\frac{\alpha_i}{s}\right)^2\|(\mathbf{u}-\mathbf{v})(P_j)\|^2 \\
&= \sum_{j\in\mathfrak{P}}\sum_{i\in\mathfrak{C}_j}\left(\frac{\alpha_i}{s}\right)^2\|(\mathbf{u}-\mathbf{v})(P_j)\|^2 \\
&= \sum_{j\in\mathfrak{P}}\left[\|(\mathbf{u}-\mathbf{v})(P_j)\|^2\sum_{i\in\mathfrak{C}_j}\left(\frac{\alpha_i}{s}\right)^2\right] \\
&\le \sum_{j\in\mathfrak{P}}[\|(\mathbf{u}-\mathbf{v})(P_j)\|^2]\max_{j\in\mathfrak{P}}\left\{\sum_{i\in\mathfrak{C}_j}\left(\frac{\alpha_i}{s}\right)^2\right\} \\
&\le \|(\mathbf{u}-\mathbf{v})\|^2\max_{j\in\mathfrak{P}}\left\{\sum_{i\in\mathfrak{C}_j}\left(\frac{\alpha_i}{s}\right)^2\right\}.
\end{aligned}$$

Taking the square root yields,

$$\|T(\mathbf{u}) - T(\mathbf{v})\| \le \|(\mathbf{u}-\mathbf{v})\|\max_{j\in\mathfrak{P}}\left\{\sqrt{\sum_{i\in\mathfrak{C}_j}\left(\frac{\alpha_i}{s}\right)^2}\right\}. \qquad (6)$$

Therefore the condition $\max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\} < 1$ satisfies the contractivity of T. This completes the proof of sufficiency. To prove the necessity, assume that T is a contraction. Hence, there exists $c < 1$ such that

$$\forall\mathbf{u},\mathbf{v}\in\mathbb{R}^{N\times N},\quad \|T(\mathbf{u}) - T(\mathbf{v})\| \le c\|\mathbf{u}-\mathbf{v}\|. \qquad (7)$$

For any $j \in \mathfrak{P}$ take the pair of images $\mathbf{u} = \mathbf{u}^{(j)}$ and $\mathbf{v} = \mathbf{v}^{(j)}$ defined in the following manner. $\mathbf{u}^{(j)}(x,y) = 1_{P_j}(x,y)$ for any $(x,y) \in \Upsilon$, where 1_{P_j} represents the characteristic function on P_j. Also, take $\mathbf{v}^{(j)}(x,y) = 0$ for any $(x,y) \in \Upsilon$. Using (5),

$$\begin{aligned}
\|T(\mathbf{u}^{(j)}) - T(\mathbf{v}^{(j)})\|^2 &= \sum_{i\in\mathfrak{C}}\alpha_i{}^2\,\big\|\mathcal{D}\big((1_{P_j} - 0)(P_{J(i)})\big)\big\|^2 \\
&= \sum_{i\in\mathfrak{C}}\alpha_i{}^2\big\|\mathcal{D}\big((1_{P_j})(P_{J(i)})\big)\big\|^2.
\end{aligned}$$

Breaking the sum $i \in \mathfrak{C}$ over two disjoint sets $i \in \mathfrak{C}$, $J(i) = j$ and $i \in \mathfrak{C}$, $J(i) \ne j$ yields

$$\begin{aligned}
&\|T(\mathbf{u}^{(j)}) - T(\mathbf{v}^{(j)})\|^2 \\
&\quad= \sum_{i\in\mathfrak{C},J(i)=j}\alpha_i{}^2\big\|\mathcal{D}\big((1_{P_j})(P_{J(i)})\big)\big\|^2 \\
&\qquad+ \sum_{i\in\mathfrak{C},J(i)\ne j}\alpha_i{}^2\big\|\mathcal{D}\big((1_{P_j})(P_{J(i)})\big)\big\|^2 \\
&\quad= \sum_{i\in\mathfrak{C},J(i)=j}\alpha_i{}^2\big\|\mathcal{D}\big(1_{P_j}\big)\big\|^2 = \sum_{i\in\mathfrak{C},J(i)=j}\alpha_i{}^2\frac{1}{s^2}\|1_{P_j}\|^2 \\
&\quad= \|1_{P_j}\|^2\sum_{i\in\mathfrak{C},J(i)=j}\left(\frac{\alpha_i}{s}\right)^2 = \|1_{P_j}\|^2\sum_{i\in\mathfrak{C}_j}\left(\frac{\alpha_i}{s}\right)^2.
\end{aligned}$$

Also, $\|\mathbf{u}^{(j)} - \mathbf{v}^{(j)}\| = \|1_{P_j} - 0\| = \|1_{P_j}\|$. Hence for any $j \in \mathfrak{P}$, replacing $\mathbf{u} = \mathbf{u}^{(j)}$ and $\mathbf{v} = \mathbf{v}^{(j)}$ in (7) gives $\|T(\mathbf{u}^{(j)}) - T(\mathbf{v}^{(j)})\| \le c\|\mathbf{u}^{(j)} - \mathbf{v}^{(j)}\|$, and replacing the equivalent values we just found yields $\forall j \in \mathfrak{P}$, $\|1_{P_j}\|\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2} \le c\|1_{P_j}\|$, for some $c < 1$. Cancelling the positive value $\|1_{P_j}\|$ from sides of the inequality yields

$$\sqrt{\sum_{i\in\mathfrak{C}_j}\left(\frac{\alpha_i}{s}\right)^2} \le c < 1, \tag{8}$$

for any $j \in \mathfrak{P}$. Hence, $\max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\} < 1$. This completes the proof of necessity. Finally to prove $con_{(T)} = \max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\}$, assume that T is a contraction on $\mathbb{R}^{N\times N}$ and $con_{(T)}$ is defined as the infimum taken over all of the possible contractivity factors of T. Because T is a contraction, the necessity condition we just proved satisfies $\max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\} < 1$. Now, we know this inequality holds and observe that (6) states $\|T(\mathbf{u}) - T(\mathbf{v})\| \le \|(\mathbf{u} - \mathbf{v})\| \max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\}$. This shows that the expression $\max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\}$ is a contractivity factor of T, and hence $con_{(T)}$ which is the infimum over all possible contractivity factor of T satisfies $con_{(T)} \le \max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\}$. It is well-known that the infimum of all contractivity factors of T, denoted by $con_{(T)}$, is itself a contractivity factor of T. Hence, taking $c = con_{(T)}$ in (8) gives $\max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\} \le con_{(T)}$. Therefore, $con_{(T)} = \max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\} < 1$. This shows that the smallest possible contractivity factor of T is $con_{(T)} = \max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\}$, which will be referred to as *the contractivity factor of* T. □

Corollary 1 *A sufficient condition for contractivity of* T *on* $\mathbb{R}^{N\times N}$ *is* $\alpha_{\max} < \frac{s}{\sqrt{\mathfrak{C}_{\max}}}$ *where* $\alpha_{\max} = \max_{i\in\mathfrak{C}} |\alpha_i|$ *and* $\mathfrak{C}_{\max} = \max_{j\in\mathfrak{P}} |\mathfrak{C}_j|$.

Proof Assuming $\alpha_{\max} < \frac{s}{\sqrt{\mathfrak{C}_{\max}}}$ yields

$$\sqrt{\sum_{i\in\mathfrak{C}_j}\left(\frac{\alpha_i}{s}\right)^2} \le \sqrt{\sum_{i\in\mathfrak{C}_j}\left(\frac{\alpha_{\max}}{s}\right)^2} = \left(\frac{\alpha_{\max}}{s}\right)\sqrt{|\mathfrak{C}_j|}$$
$$< \left(\frac{s}{\sqrt{\mathfrak{C}_{\max}}}\frac{1}{s}\right)\sqrt{|\mathfrak{C}_j|} < \frac{\sqrt{|\mathfrak{C}_j|}}{\sqrt{|\mathfrak{C}_{\max}|}} \le 1.$$

Therefore, the sufficient contractivity condition of Theorem 1 is satisfied. □

Note that $\mathfrak{C}_{\max}$ is the maximum number of times any single domain block is mapped to some range block. In the worst case, when all of the range blocks are mapped from a single domain block, $\mathfrak{C}_{\max}$ would be equal to the number of all range blocks, or $|\mathfrak{C}|$. This leads to $\alpha_{\max} < \frac{s}{\sqrt{\mathfrak{C}_{\max}}} = \frac{s}{\sqrt{|\mathfrak{C}|}}$. However, in a typical situation where each domain block is related to only a few range blocks $\mathfrak{C}_{\max}$ is small and the α_is can be relaxed and take larger values in magnitude up to $\frac{s}{\sqrt{\mathfrak{C}_{\max}}}$.

Corollary 2 T *is a contraction on* $\mathbb{R}^{N\times N}$ *if* $\sqrt{\sum_{i\in\mathfrak{C}}(\frac{\alpha_i}{s})^2} < 1$.

Proof $\mathfrak{C}_j \subseteq \mathfrak{C}, \forall j \in \mathfrak{P}$. Hence, $\max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\} \le \sqrt{\sum_{i\in\mathfrak{C}}(\frac{\alpha_i}{s})^2} < 1$, which satisfies the sufficient contractivity condition given in Theorem 1. □

This simple corollary we just proved gives sufficient contractivity condition of T on $\mathbb{R}^{N\times N}$ and has the same form of the sufficient condition given in [9] for the continuous case. Theorem 1 that we proved earlier was a stronger statement, i.e., a more relaxed sufficient condition for the contractivity of T.

4 Convergence Results for the Fractal Transform Operator

The contractivity of T is sufficient for the convergence of the fractal decoding scheme. It is, however, not always necessary. In this section, we derive analytical results on the convergence of the fractal decoding. The following simple example shows that the fractal decoding scheme may be convergent to a unique limit independent of the starting point even when T is not a contraction.

Example 1 Consider an image $\mathbf{u} = \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$ containing only 4 pixels with the corresponding intensities in the matrix, and a simple associated fractal transform operator T defined in the following way. The only 2×2 domain block equals the whole image $\mathbf{u}$ and the four range blocks are of size 1×1 corresponding to the four pixels of the image. Hence, a decimation factor of $s = 2$ is considered. Assume the α_i values corresponding to a mapping of the decimated domain block (the whole image) to every range block (pixel) are summarized in the matrix $\left[\begin{smallmatrix}3 & 0\\ 0 & 0\end{smallmatrix}\right]$ and the associated β_i values are given by $\left[\begin{smallmatrix}1/4 & 0\\ 0 & 0\end{smallmatrix}\right]$. Hence, for any $\mathbf{x} = \left[\begin{smallmatrix}x_1 & x_2\\ x_3 & x_4\end{smallmatrix}\right]$, $T(\mathbf{x}) = T(\left[\begin{smallmatrix}x_1 & x_2\\ x_3 & x_4\end{smallmatrix}\right]) = \left[\begin{smallmatrix}\frac{1+3(x_1+x_2+x_3+x_4)}{4} & 0\\ 0 & 0\end{smallmatrix}\right]$. It is easy to verify that $\mathbf{u}$ is the unique fixed point of T and $T(\mathbf{u}) = \mathbf{u}$. However, $\max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\} = \sqrt{(\frac{3}{2})^2} = 1.5 > 1$ and T is not a contraction on $\mathbb{R}^{2\times 2}$ with the Frobenius norm. One can also directly verify that T is not a contraction, e.g.,

$$\left\| T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \right\| = \left\| \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \right\| = 3 > \left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\| = 2.$$

However, it can be easily verified that independent of $\mathbf{u}_0$ the iterative sequence of images $\mathbf{u}_{n+1} = T(\mathbf{u}_n)$ converges to $\mathbf{u}$.

It is now understood that all fractal transforms will be written in matrix notation. Given T we can define a corresponding $\underline{T}$ acting on $\underline{\mathbf{x}}$ such that $\underline{T}(\underline{\mathbf{x}})$ is $T(\mathbf{x})$ represented in the vector format. We also take advantage of writing $\underline{\mathbf{x}}$ in the vector format, so that operator $\underline{T}$ can be written as $\underline{T}(\underline{\mathbf{x}}) = \mathbf{M}\underline{\mathbf{x}} + \mathbf{B}$. Such matrix representation of the fractal transform operator has been introduced in [13]. Here, $\mathbf{M}$ is an $N^2 \times N^2$ matrix and both $\underline{\mathbf{x}}$ and $\mathbf{B}$ are vectors of dimension $N^2 \times 1$. Matrix $\mathbf{M}$ carries the α_i information and also depends on s. $\mathbf{B}$ contains the information of β_i, all in the appropriate locations. Recall that α_i and β_i were the parameters of the greyscale maps. We are seeking conditions under which the iterations $\mathbf{u}_{n+1} = T(\mathbf{u}_n)$ converge independent of the initial arbitrary image $\mathbf{u}_0$. In the vector-matrix notation, the iterative scheme reads

$$\underline{\mathbf{u}}_{n+1} = \mathbf{M}\underline{\mathbf{u}}_n + \mathbf{B}. \tag{9}$$

The following proposition proved in [13] gives a useful sufficient condition for the convergence of the fractal decoding scheme.

Proposition 1 [13] *A sufficient condition for the convergence of the fractal decoding scheme* $\mathbf{u}_{n+1} = T(\mathbf{u}_n)$ *independent of the starting image* $\mathbf{u}_0$ *is* $|\alpha_i| < 1, \forall i \in \mathfrak{C}$.

Although this sufficient condition may be useful in some cases, it is not necessary for the convergence of the fractal decoding scheme. Example 1 is a case for which the fractal decoding scheme converges independent of the starting image, yet there exists a value of $\alpha_i = 3 > 1$ and the sufficient condition of Proposition 1 is not satisfied. The convergence of the fractal decoding scheme to a limit independent of the starting image is given by the condition $\rho(\mathbf{M}) < 1$, (where $\rho(\mathbf{M})$ is the spectral radius of $\mathbf{M}$) based on the following well-known proposition.

Proposition 2 [13, 15, 16] *The linear iterative scheme in* (9) *converges to a limit independent of* $\underline{\mathbf{u}}_0$ *if and only if* $\rho(\mathbf{M}) < 1$.

Finally, the fact that the fractal decoding process may converge to a limit, even if the condition $\max_{j\in\mathfrak{P}}\{\sqrt{\sum_{i\in\mathfrak{C}_j}(\frac{\alpha_i}{s})^2}\} < 1$ is not satisfied, is due to $\rho(\mathbf{M}) < 1$ for the associated fractal code. We can generalize the Collage Theorem (4) to include cases for which the fractal decoding scheme is converging to a limit while T is not a contraction.

Theorem 2 (The Generalized Collage Theorem) *Assume the iterative sequence* $\mathbf{u}_{n+1} = T(\mathbf{u}_n)$ *converges to* $\mathbf{u}^\star$ *independent of* $\mathbf{u}_0$. *Then*

$$\|\mathbf{u} - \mathbf{u}^\star\| \le \|\mathbf{u} - T(\mathbf{u})\| \cdot \|(\mathbf{I}_{N^2\times N^2} - \mathbf{M})^{-1}\|_2, \tag{10}$$

where $\|.\|_2$ *represents the matrix* 2*-norm.*

Proof Considering the fractal transform operator in matrix notation, and using induction for any $k \in \mathbb{N}$, yields $\underline{T}^k(\underline{\mathbf{u}}) = \mathbf{M}^k\underline{\mathbf{u}} + (\sum_{m=0}^{k-1}\mathbf{M}^m)\mathbf{B}$. Hence for any $k \in \mathbb{N}$

$$\begin{aligned}\|\underline{\mathbf{u}} - \underline{T}^k(\underline{\mathbf{u}})\| &= \left\| \underline{\mathbf{u}} - \mathbf{M}^k\underline{\mathbf{u}} - \left(\sum_{m=0}^{k-1}\mathbf{M}^m\right)\mathbf{B} \right\| \\ &= \left\| \sum_{m=0}^{k-1}(\mathbf{M}^m - \mathbf{M}^{m+1})\underline{\mathbf{u}} - \left(\sum_{m=0}^{k-1}\mathbf{M}^m\right)\mathbf{B} \right\| \\ &= \left\| \sum_{m=0}^{k-1}\mathbf{M}^m\left[(\mathbf{I}_{N^2\times N^2} - \mathbf{M})\underline{\mathbf{u}} - \mathbf{B}\right] \right\| \\ &\le \|(\mathbf{I}_{N^2\times N^2} - \mathbf{M})\underline{\mathbf{u}} - \mathbf{B}\| \cdot \left\| \sum_{m=0}^{k-1}\mathbf{M}^m \right\|_2.\end{aligned}$$

The assumption that the sequence $\mathbf{u}_n$ converges to $\mathbf{u}^\star$ independent of $\mathbf{u}_0$ is equivalent to $\rho(\mathbf{M}) < 1$. If $\rho(\mathbf{M}) < 1$, the limit of $\sum_{m=0}^{k-1}\mathbf{M}^m$ as $k \to \infty$ exists and equals $(\mathbf{I}_{N^2\times N^2} - \mathbf{M})^{-1}$ [15]. Hence, taking the limit $k \to \infty$ and converting to the non-vector notation yields

$$\|\mathbf{u} - \mathbf{u}^\star\| \le \|\mathbf{u} - T(\mathbf{u})\| \cdot \|(\mathbf{I}_{N^2\times N^2} - \mathbf{M})^{-1}\|_2. \tag{11}$$

We can also prove that (11) implies the Collage Theorem in (4). First we show that if T is a contraction with contractivity factor $con_{(T)}$, $0 \le con_{(T)} < 1$, then $\|\mathbf{M}\|_2 \le con_{(T)}$. Note that for any $\underline{\mathbf{x}} \ne \underline{\mathbf{0}}$

$$\begin{aligned}\|\mathbf{M}\underline{\mathbf{x}}\| &= \|\mathbf{M}\underline{\mathbf{x}} - \mathbf{M}\underline{\mathbf{0}}\| = \|\mathbf{M}\underline{\mathbf{x}} + \mathbf{B} - (\mathbf{M}\underline{\mathbf{0}} + \mathbf{B})\| \\ &= \|\underline{T}(\underline{\mathbf{x}}) - \underline{T}(\underline{\mathbf{0}})\| \le con_{(T)}\|\underline{\mathbf{x}}\|.\end{aligned}$$

Therefore, $\forall\underline{\mathbf{x}} \ne \underline{\mathbf{0}}$, $\frac{\|\mathbf{M}\underline{\mathbf{x}}\|}{\|\underline{\mathbf{x}}\|} \le con_{(T)}$, and $\|\mathbf{M}\|_2 = \sup_{\underline{\mathbf{x}}\ne\underline{\mathbf{0}}}\frac{\|\mathbf{M}\underline{\mathbf{x}}\|}{\|\underline{\mathbf{x}}\|} \le con_{(T)}$. Hence,

$$\begin{aligned}\|(\mathbf{I}_{N^2\times N^2} - \mathbf{M})^{-1}\|_2 &= \left\| \sum_{m=0}^{\infty}\mathbf{M}^m \right\|_2 \le \sum_{m=0}^{\infty}\|\mathbf{M}\|_2^m \\ &\le \sum_{m=0}^{\infty}con_{(T)}^m = \frac{1}{1 - con_{(T)}}.\end{aligned}$$

Using this inequality and (11) yields the Collage Theorem

$$\|\mathbf{u} - \mathbf{u}^\star\| \le \frac{1}{1 - con_{(T)}} \cdot \|\mathbf{u} - T(\mathbf{u})\|. \qquad \square$$

To examine the derived results for the fractal transform operator T of Example 1, note that the corresponding

$$\mathbf{M} = \frac{3}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \rho(\mathbf{M}) = 3/4 < 1.$$

Hence, Proposition 2 guarantees the convergence of the iterations of T for any arbitrary starting image. Therefore, the conditions of Theorem 2 are satisfied. Furthermore,

$$\|(\mathbf{I}_{4\times 4} - \mathbf{M})^{-1}\|_2 = \left\| \begin{bmatrix} 4 & 3 & 3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\|_2 = \sqrt{22 + 6\sqrt{13}}$$

which can be applied in (10). Both sides of (10) are zero for the given $\mathbf{u}$ and T of Example 1, because $\mathbf{u}$ was the unique fixed-point of T.

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Mehran Ebrahimi received his Ph.D. in Applied Mathematics from the University of Waterloo, Canada, in 2008. He is currently a postdoctoral fellow at the Department of Medical Biophysics, University of Toronto, Canada, at the Sunnybrook Health Sciences Centre. His research is supported in part by a Natural Sciences and Engineering Research Council of Canada (NSERC) Postdoctoral Fellowship (PDF) award. His current research interests include inverse problems in imaging sciences.