

COUPLED MULTI-FRAME SUPER-RESOLUTION WITH DIFFUSIVE MOTION MODEL AND TOTAL VARIATION REGULARIZATION

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ABSTRACT

The problem of recovering a high-resolution image from a set of distorted (e.g., warped, blurred, noisy) and low-resolution images is known as *super-resolution*.

Accurate motion estimation from low-resolution measurements is a fundamental challenge of the super-resolution problem. Some recent promising advances in this area have been focused on coupling or combining the super-resolution reconstruction and the motion estimation. However, the existing approaches are limited to parametric motion models, e.g., affine transformations. In this paper, we shall address the coupled super-resolution problem with a *non-parametric* motion model. We then consider a variational formulation of the problem and use a PDE-approach to construct a numerical scheme for its solution. In this paper, diffusion regularization is used for the motion model and total variation regularization for the super-resolved image.

1. INTRODUCTION

Naturally, there is always a demand for higher quality and higher resolution images. The level of image detail is crucial for the performance of many computer vision algorithms [3, 4, 5, 6, 7, 8, 9, 10, 11]. Many of the current imaging devices typically consist of arrays of light detectors. A detector determines pixel intensity values depending upon the amount of light detected from its assigned area in the scene. The spatial resolution of images produced is proportional to the density of the detector array: the greater the number of pixels in the image, the higher the spatial resolution [3]. In many applications, however, the imaging sensors have poor resolution output. When resolution can not be improved by replacing sensors, either because of cost or hardware physical limits, one can resort to resolution enhancement algorithms. Even when superior equipment is available, such algorithms provide an inexpensive alternative. The problem of recovering a high-resolution (HR) image from a set of distorted (e.g., warped, blurred, noisy) and low-resolution (LR) images is

known as *super-resolution* [7, 8, 9, 10, 4, 11, 3].

Fusion of the information from the observations is a fundamental challenge in the recovery process. With just one imaging device and under the same lighting conditions, we require some relative motions from frame to frame. Each LR frame should provide a different view of the same scene. Motion and nonredundant information obtained from different frames are what make super-resolution feasible [3].

1.1. A Brief History

The super-resolution literature has significantly expanded in the past 20 years. A rather recent and comprehensive survey of super-resolution techniques is given in [4]. Historically, Irani and Peleg [9] proposed an iterative back-projection method to address the super-resolution problem. Sauer and Allebach [12], and Tekalp, Ozkan and Sezan [13] modelled super-resolution as an interpolation problem with nonuniformly sampled data and used a projection onto convex sets algorithm to reconstruct the image. Ur and Gross [14] considered Papoulis' generalized multichannel sampling theorem [15] for interpolating values on a higher resolution grid. Shekarforoush and Chellappa [16] extended Papoulis' theorem for merging the nonuniform samples of multiple channels into HR data. Aizawa *et al.* [17] also modelled super-resolution as an interpolation problem with nonuniform sampling and used a formula related to Shannon's sampling theorem [18] to estimate values on a HR grid. Tsai and Huang [19] were among the first to superresolve a HR image from several sampled LR frames. Hardie *et al.* [8] proposed a joint MAP registration and restoration algorithm using a *Gibbs image prior*. Schultz and Stevenson [10] used a Markov random field model with Gibbs prior to better represent image discontinuities, such as transitions across sharp edges. More recently Farsiu *et al.* [5] proposed an alternative data fidelity, or regularization term based on the ℓ^1 norm which has been shown to be robust to data outliers. They proposed a novel regularization term

called ‘‘Bilateral-TV’’ which provides robust performance while preserving the edge content common to real image sequences.

1.2. Coupled Motion Estimation

Accurate motion estimation has been a very important aspect of super-resolution schemes. In many existing super-resolution approaches, the motion is computed directly from the LR frames, while many other super-resolution algorithms unrealistically assume that motion parameters are precisely known. In general, however, accurate motion estimation of subpixel accuracy remains a fundamental challenge in super-resolution reconstruction algorithms.

In a recent work [2], it has been suggested that the motion can be relaxed from a strict grid mapping to a multi-pixel-pair intensity relation. In this view, pixel-pairs in different frames may be relevant to each other with some measured probability of confidence. Instead of estimating the motion vectors explicitly, a framework is provided in which such confidence measures are evaluated and employed in the HR image reconstruction. The algorithm is computationally intensive, however.

In general, it is believed a combined super-resolution reconstruction and motion estimation may be the key to address the super-resolution problem. A novel method towards this direction is proposed in [20]. Although the authors of [20] appreciate the importance of considering non-parametric motion models, their proposed method is restricted to the parametric affine motion model. Their preference to work with a parametric motion model lies in the complexity of formulations of the non-parametric approaches as they discuss in their paper.

1.3. The Agenda

In this work, we propose the coupled multi-frame super-resolution problem with a *non-parametric* motion model, namely the method of diffusion registration [21] (Chapter 11). It is so named because the resulting PDE may be viewed as a generalized diffusion equation. In Section 2, we cast the super-resolution problem as a minimization problem and then present its corresponding variational formulation. For consistency, all notation is adopted from [21]. In Section 3, we derive a PDE with a steady-state solution that corresponds to the solution of the described problem. The discretization and derivation of a numerical scheme for the PDE is presented in Section 4. Finally, we present the results of some computational experiments Section 5, along with concluding remarks in Section 6.

2. MATHEMATICAL FORMULATION

Throughout, images are d -dimensional and are assumed as compactly supported elements of $\mathcal{L}^2(\Omega)$, $\Omega \subset \mathbb{R}^d$, unless otherwise stated.

Forward Model. Assume that m low-resolution measurement images y_1, y_2, \dots, y_m of an ideal image f are given.

For every $i = 1, \dots, m$, y_i is a noisy, low-resolution realization of deformed copies of f via a d -dimensional vector field $u_i = (u_{i,1}, \dots, u_{i,d})$. Namely,

$$y_i := \mathcal{H}f_{u_i} + n_i, \quad i = 1, \dots, m, \quad (1)$$

where n_i is the additive noise, and f_{u_i} denotes the deformed image f via u_i , i.e., $f_{u_i}(x) = f(x - u_i(x))$.

Throughout, we may also use the alternative notation of

$$\mathcal{S}_i f := f_{u_i}.$$

Note that the operator \mathcal{S}_i is linear with respect to f although f_{u_i} is a nonlinear expression with respect to u_i . The operator $\mathcal{H} : \mathcal{L}^2(\Omega) \rightarrow \mathcal{L}^2(\Omega)$ is assumed to be a known linear degradation operator modeled as a composition of a spatially invariant blur \mathcal{K} followed by a down-sampling operator \mathcal{D} , i.e., $\mathcal{H} := \mathcal{D} \circ \mathcal{K}$. Here, \mathcal{D} is an impulse train constructed using the sum of uniformly spaced Dirac functions [3, 4, 5, 6, 7]. To proceed, we formulate the corresponding super-resolution problem. As opposed to what is typically done in the literature, we assume that both the deformations and the high-resolution image are unknown and try to recover both simultaneously.

Problem 1. *Given a set of m low-resolution measured images represented by $y := \{y_1, y_2, \dots, y_m\}$ and a degradation operator \mathcal{H} , find a corresponding set of deformations $u := \{u_1, u_2, \dots, u_m\}$ and a high-resolution image f that minimizes*

$$\mathcal{J}[u, f] := \mathcal{C}[y; (u, f)] + \mathcal{R}[u, f]$$

in which \mathcal{C} measures the consistency of the measurements y with the high-resolution image f , and \mathcal{R} is a regularization expression on $[u, f]$. Here, we use the sum of squares of intensity differences for the consistency measure

$$\mathcal{C}[y; (u, f)] := \frac{1}{2} \sum_{i=1}^m \|y_i - \mathcal{H}f_{u_i}\|_{\mathcal{L}^2(\Omega)}^2, \quad (2)$$

and the regularization is defined by

$$\mathcal{R}[u, f] := \sum_{i=1}^m \alpha_i \mathcal{P}[u_i] + \beta \mathcal{Q}[f]. \quad (3)$$

Here, $\alpha_1, \dots, \alpha_m, \beta \in \mathbb{R}^+$ are positive regularizing parameters and \mathcal{P} and \mathcal{Q} represent regularizations on u_i and f respectively. Hence, the objective is to minimize

$$\mathcal{J}[u, f] = \frac{1}{2} \sum_{i=1}^m \|y_i - \mathcal{H}f_{u_i}\|_{\mathcal{L}^2(\Omega)}^2 + \sum_{i=1}^m \alpha_i \mathcal{P}[u_i] + \beta \mathcal{Q}[f]. \quad (4)$$

We shall present a mathematical formulation to solve Problem 1. Briefly, we seek necessary conditions for optimality of $[u, f]$ by finding the Gâteaux derivatives of the components of \mathcal{J} with respect to $[u, f]$. This shall provide us with the corresponding Euler-Lagrange equations that will be used to form a PDE which will be solved numerically.

Theorem 1. Let $d \in \mathbb{N}$, and assume that f, y_1, y_2, \dots, y_m are d -dimensional real-valued images, i.e., functions from $\Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$, $f \in C^2(\mathbb{R}^d)$, $u_1, \dots, u_m : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $v : \mathbb{R}^d \rightarrow \mathbb{R}^{md+1}$, $\Omega :=]0, n[^d$. The Gâteaux derivative of $\mathcal{C}[y; (u, f)]$ is given by

$$d\mathcal{C}[y; (u, f); v] = - \int_{\Omega} \langle \Phi(x, u(x), f(x)), v(x) \rangle_{\mathbb{R}^{md+1}} dx,$$

in which $\Phi : \mathbb{R}^d \times \mathbb{R}^{md} \times \mathbb{R} \rightarrow \mathbb{R}^{md+1}$,

$$\Phi(x, u(x), f(x)) = [p_1(x), \dots, p_m(x), q(x)],$$

where

$$\begin{aligned} p_i(x) &:= \mathcal{H}^*[\mathcal{H}\mathcal{S}_i f(x) - y_i(x)] \nabla \mathcal{S}_i f(x), \\ &\quad i = 1, \dots, m, \\ q(x) &:= - \sum_{i=1}^m \mathcal{S}_i^* \mathcal{H}^*[\mathcal{H}\mathcal{S}_i f(x) - y_i(x)], \end{aligned}$$

in which \mathcal{H}^* and \mathcal{S}_i^* represent the adjoint of operators \mathcal{H} and \mathcal{S}_i respectively.

Proof. See the proof in Appendix 1 [Cf. [21] pp. 80]. \square

Here, we focus on the special case where \mathcal{P} is the diffusion regularization [21, 22, 23, 24, 25, 26], and \mathcal{Q} the total variation regularization functional [27, 28].

Theorem 2. Assume \mathcal{P} is the diffusion regularization functional and \mathcal{Q} is the total variation regularization expression and the functionals \mathcal{P}_i^e and \mathcal{Q}^e are respectively trivial extensions of \mathcal{P} and \mathcal{Q} , i.e.,

$$\begin{aligned} \mathcal{P}_i^e[(u, f)] &:= \mathcal{P}[u_i] := \frac{1}{2} \sum_{j=1}^d \int_{\Omega} \langle \nabla u_{i,j}, \nabla u_{i,j} \rangle dx, \\ &\quad i = 1, \dots, m, \end{aligned} \quad (5)$$

$$\mathcal{Q}^e[(u, f)] := \mathcal{Q}[f] :=$$

$$TV[f] = \int_{\Omega} |\nabla f(x)|_{\epsilon} dx = \int_{\Omega} \sqrt{|\nabla f(x)|^2 + \epsilon} dx, \quad (6)$$

for some positive constant ϵ . Also, assume that Neumann boundary conditions are imposed, i.e.,

$$\langle \nabla f(x), \vec{n}(x) \rangle_{\mathbb{R}^d} = \langle \nabla u_{i,j}(x), \vec{n}(x) \rangle_{\mathbb{R}^d} = 0$$

for $x \in \partial\Omega$ and $j = 1, \dots, d$, in which \vec{n} denotes the outer normal unit vector of $\partial\Omega$ (boundary of Ω). The Gâteaux derivatives of $\mathcal{P}_i^e[(u, f); v]$ and $\mathcal{Q}^e[(u, f); v]$ are, respectively,

$$\begin{aligned} d\mathcal{P}_i^e[(u, f); v] &= - \int_{\Omega} \langle \mathcal{A}_i[u](x), v(x) \rangle_{\mathbb{R}^{d+1}} dx, \\ &\quad i = 1, \dots, m, \end{aligned}$$

$$d\mathcal{Q}^e[(u, f); v] = - \int_{\Omega} \langle \mathcal{B}[f](x), v(x) \rangle_{\mathbb{R}^{d+1}} dx,$$

where,

$$\mathcal{A}_i[u](x) =$$

$$\begin{aligned} &(\underbrace{0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d}}_{i-1 \text{ times}}, \Delta u_{i,1}(x), \dots, \Delta u_{i,d}(x), \underbrace{0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d}}_{m-i \text{ times}}, 0) \\ &= (\underbrace{0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d}}_{i-1 \text{ times}}, \Delta u_i(x), \underbrace{0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d}}_{m-i \text{ times}}, 0), \end{aligned}$$

$$\begin{aligned} \mathcal{B}[f](x) &= (\underbrace{0_{\mathbb{R}^d}, \dots, 0_{\mathbb{R}^d}}_{m \text{ times}}, \nabla \cdot \left(\frac{\nabla f(x)}{|\nabla f(x)|_{\epsilon}} \right)) \\ &= (0_{\mathbb{R}^{md}}, \nabla \cdot \left(\frac{\nabla f(x)}{|\nabla f(x)|_{\epsilon}} \right)). \end{aligned}$$

Proof. The result corresponding to $\mathcal{A}_i[u](x)$ yields applying the Green's formula similar to [21] pp. 138. See, e.g. [28], for the second part relating to $\mathcal{B}[f](x)$. \square

Theorem 3. The Euler-Lagrange equations corresponding to the objective function \mathcal{J} have the form,

$$\Phi(x, u(x), f(x)) + \sum_{i=1}^m \alpha_i \mathcal{A}_i[u](x) + \beta \mathcal{B}[f](x) = 0, \quad (7)$$

for all $x \in \Omega$ with Neumann boundary conditions. These equations can also be written as follows,

$$\begin{aligned} \mathcal{H}^*[\mathcal{H}\mathcal{S}_i f(x) - y_i(x)] \nabla \mathcal{S}_i f(x) + \alpha_i \Delta u_i(x) &= 0_{\mathbb{R}^d}, \\ &\quad i = 1, \dots, m, \quad x \in \Omega, \end{aligned}$$

$$\begin{aligned} - \sum_{i=1}^m \mathcal{S}_i^* \mathcal{H}^*[\mathcal{H}\mathcal{S}_i f(x) - y_i(x)] + \beta \nabla \cdot \left(\frac{\nabla f(x)}{|\nabla f(x)|_{\epsilon}} \right) &= 0, \\ &\quad x \in \Omega, \end{aligned}$$

$$\begin{aligned} \langle \nabla f(x), \vec{n}(x) \rangle_{\mathbb{R}^d} = \langle \nabla u_{i,j}(x), \vec{n}(x) \rangle_{\mathbb{R}^d} &= 0, \\ &\quad j = 1, \dots, d, \quad i = 1, \dots, m, \quad x \in \partial\Omega. \end{aligned}$$

Proof. The result yields from Theorem 1 and substitution (Cf. [21] pp. 138). \square

3. A CORRESPONDING PDE

There exist various ways to solve Equation (7). A possibility that we pursue here is to formulate the solution as the steady-state solution of a corresponding PDE similar to what is one in [21] (pp 80). We propose

$$\begin{aligned} \partial_t(u(x, t), s f(x, t)) &= \Phi(x, u(x, t), f(x, t)) \\ &+ \sum_{i=1}^m \alpha_i \mathcal{A}_i[u](x) + \beta \mathcal{B}[f](x) \quad x \in \Omega, \quad t \geq 0, \end{aligned}$$

where s is a scale factor. Assuming $\Phi = (p_1, \dots, p_m, q)$ the PDE can be written as

$$\partial_t u_i(x, t) = p_i(x, u(x, t), f(x, t)) + \alpha_i \Delta u_i(x, t), \quad (8)$$

$$i = 1, \dots, m, \quad x \in \Omega, \quad t \geq 0,$$

$$\begin{aligned} s \partial_t f(x, t) &= q(x, u(x, t), f(x, t)) + \beta \nabla \cdot \left(\frac{\nabla f(x, t)}{|\nabla f(x, t)|_{\epsilon}} \right), \\ &\quad x \in \Omega, \quad t \geq 0, \end{aligned} \quad (9)$$

where,

$$p_i(x, u, f) := \mathcal{H}^*[\mathcal{H}\mathcal{S}_i f(x) - y_i(x)] \nabla \mathcal{S}_i f(x),$$

$$i = 1, \dots, m,$$

$$q(x, u, f) := - \sum_{i=1}^m \mathcal{S}_i^* \mathcal{H}^*[\mathcal{H}\mathcal{S}_i f(x) - y_i(x)].$$

4. DISCRETIZATION AND NUMERICAL SCHEME

To numerically solve the derived PDE in Equations (8,9), we evaluate expressions at discrete time variable $\{t_{k+1}\}$

$$\begin{aligned} \partial_t u_i(x, t_{k+1}) &= p_i(x, u(x, t_k), f(x, t_k)) \\ &\quad + \alpha_i \Delta u_i(x, t_{k+1}), \end{aligned} \quad (10)$$

$$i = 1, \dots, m, \quad x \in \Omega,$$

$$\begin{aligned} s \partial_t f(x, t_k) &= q(x, u(x, t_k), f(x, t_k)) \\ &\quad + \beta \Delta f(x, t_k), \end{aligned} \quad x \in \Omega. \quad (11)$$

Notice that due to the nonlinearity of p_i with respect to u_i , f is evaluated at t_k instead of t_{k+1} in Equation (10) [cf. [21] pp. 80] which translates to applying a fixed-point iteration scheme. However, q is linear with respect to f and t_k is used consistently in Equation (11). Using a spatial discretization X of Ω that includes n^d voxels (pixels) corresponding to a unit space step in every dimension due to the definition of $\Omega :=]0, n[^d$, and a time step of τ_1 , we define for $j = 1, \dots, d$, and $k = 0, 1, 2, \dots$

$$\begin{aligned} U_{i,j}^k(X) &:= u_{i,j}(X, \tau_1 k) := \text{Discretized}(u_j(x, t_k)), \\ F^k(X) &:= f(X, \tau_1 k) := \text{Discretized}(f(x, t_k)), \\ Y_i(X) &:= y_i(X) := \text{Discretized}(y_i(x)). \end{aligned}$$

Furthermore,

$$\mathbf{A}U_{i,j}^k := \Delta u_{i,j}(X, \tau_1 k) := \text{Discretized}(\Delta u_{i,j}(x, t_k)),$$

in which $\mathbf{A} \in \mathbb{R}^{n^d \times n^d}$ is defined such that

$$\mathbf{A}U_{i,j}^k \approx \sum_{l=1}^d \partial_{x_l} u_{i,j}(X, \tau_1 k),$$

[See Appendix 2 for the precise definition of \mathbf{A}] and

$$\begin{aligned} \mathbf{L}(F^k)F^k &\approx \nabla \cdot \left(\frac{\nabla f(X, \tau_1 k)}{|\nabla f(X, \tau_1 k)|_\epsilon} \right) \\ &:= \text{Discretized} \left(\nabla \cdot \left(\frac{\nabla f(x, t_k)}{|\nabla f(x, t_k)|_\epsilon} \right) \right). \end{aligned}$$

[See [28], pp. 134 Eq. (8.26), for the precise definition of $\mathbf{L}(F^k)$]. Also, assume that \mathbf{H} is a matrix that represents \mathcal{H} , and its transpose \mathbf{H}^T represents \mathcal{H}^* in the discretization. Appendix 3 gives the precise definition of \mathbf{H} , which is assumed as the matrix product of the local averaging blur \mathbf{K} by a zooming factor of $z \in \mathbb{N}$ in every direction, multiplied by down-sampling matrix of factor z in every

direction represented by \mathbf{D} . Furthermore, \mathbf{S}_i^k is the sparse matrix constructed using U_i^k . This matrix provides a discrete approximation of the operator \mathcal{S}_i for $i = 1, \dots, m$. [See [20], Eqs. (3,4), for the precise construction of such matrix using linear interpolation for 2-dimensional images i.e., $d = 2$.]

Substituting the discretization in the PDEs of Equations (10,11) leads that for $j = 1, \dots, d$, $k = 0, 1, 2, \dots$

$$\begin{aligned} \frac{U_{i,j}^{k+1} - U_{i,j}^k}{\tau_1} &= \\ &\quad \mathbf{H}^T[\mathbf{H}\mathbf{S}_i^k F^k - Y_i] \cdot \partial_j \mathbf{S}_i^k F^k + \alpha_i \mathbf{A}U_{i,j}^{k+1}, \\ \frac{F^{k+1} - F^k}{s \tau_1} &= \\ &\quad - \sum_{i=1}^m (\mathbf{S}_i^k)^T \mathbf{H}^T[\mathbf{H}\mathbf{S}_i^k F^k - Y_i] + \beta \mathbf{L}(F^k)F^k. \end{aligned}$$

Defining $\tau_2 := \tau_1/s$ gives

$$\begin{aligned} (\mathbf{I} - \tau_1 \alpha_i \mathbf{A})U_{i,j}^{k+1} &= U_{i,j}^k + \tau_1 \mathbf{H}^T[\mathbf{H}\mathbf{S}_i^k F^k - Y_i] \cdot \partial_j \mathbf{S}_i^k F^k, \\ F^{k+1} &= F^k + \\ &\quad \tau_2 \left[- \sum_{i=1}^m (\mathbf{S}_i^k)^T \mathbf{H}^T[\mathbf{H}\mathbf{S}_i^k F^k - Y_i] + \beta \mathbf{L}(F^k)F^k \right], \end{aligned}$$

where $\mathbf{I} \in \mathbb{R}^{n^d \times n^d}$ is the identity matrix. This yields

$$\begin{aligned} U_{i,j}^{k+1} &= (\mathbf{I} - \tau_1 \alpha_i \mathbf{A})^{-1} \times \\ &\quad \left[U_{i,j}^k + \tau_1 \mathbf{H}^T[\mathbf{H}\mathbf{S}_i^k F^k - Y_i] \cdot \partial_j \mathbf{S}_i^k F^k \right], \\ F^{k+1} &= F^k + \\ &\quad \tau_2 \left[- \sum_{i=1}^m (\mathbf{S}_i^k)^T \mathbf{H}^T[\mathbf{H}\mathbf{S}_i^k F^k - Y_i] + \beta \mathbf{L}(F^k)F^k \right], \end{aligned}$$

Finally, we use the initialization vectors $F^0 = U_{i,j}^0 = 0_{\mathbb{R}^{n^d}}$, for $i = 1, \dots, m, j = 1, \dots, d$.

5. COMPUTATIONAL EXPERIMENTS

In this section, we present two computational examples to verify the derived numerical scheme. Figures 1 and 2 show the results of evaluating the proposed super-resolution algorithm on image sequences taken from the data-set library of MDSP at U. California Santa Cruz.

In Figure 1, the first $m = 30$ frames, of an 8-bit text sequence, of size 32×32 is used and independent additive white Gaussian noise of standard deviation $\sigma = 5$ is added to the LR frames. The results are shown along with the parameters described in the caption of the Figure.

In Figure 2, the same kind of experiment is performed over a total of $m = 60$ frames, of an 8-bit surveillance sequence, of size 32×32 and again independent additive white Gaussian noise of standard deviation $\sigma = 5$ is added to the LR frames. The results are shown along with the parameters described in the Figure's caption.

The efficiency of the technique can be simply observed by comparing the image (e) of each Figure to the other images in the figure.

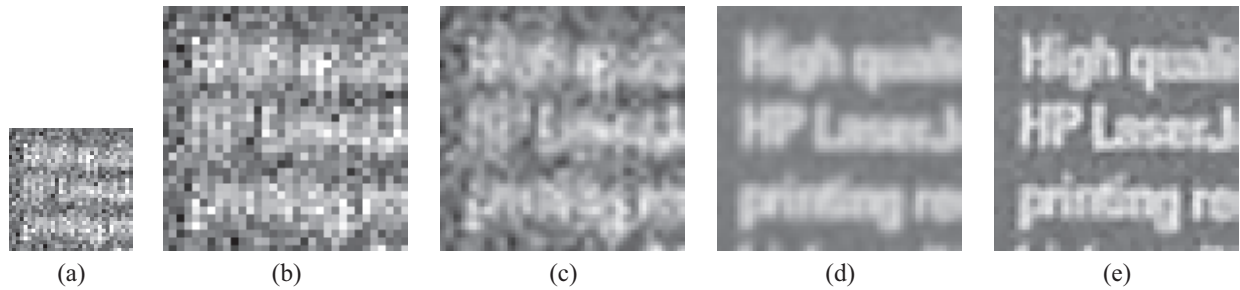


Figure 1. (a) LR frame #1. (b) Nearest neighbor interpolation on the LR frame #1. (c) Bilinear interpolation on the LR frame #1. (d) Average of the bilinear interpolation of all 60 LR frames. (e) Super resolution result of frame #1 in $d = 2$ dimensions of $n = 32$ pixels in every dimension, zooming factor $z = 2$, total number of LR frames $m = 30$, regularization parameters $\alpha_i=4000$, $i = 1, \dots, 30$, $\beta = 10$, $\epsilon = 0.1$, and time steps $\tau_1 = 0.002$, $\tau_2 = 0.02$, where 30 iterations are applied.

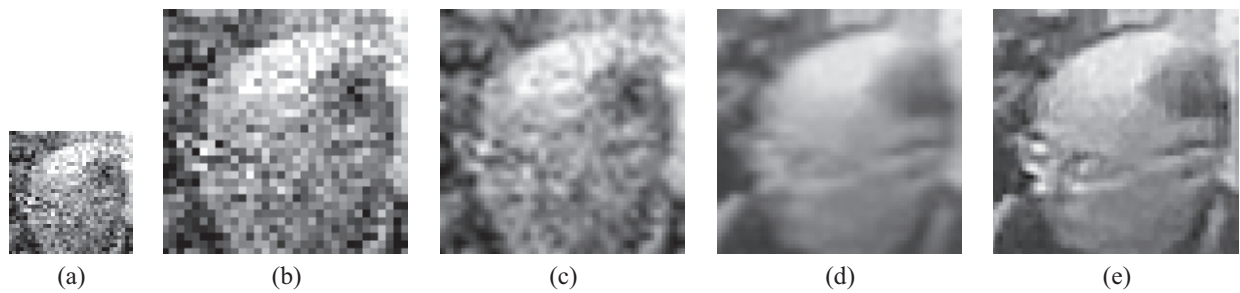


Figure 2. (a) LR frame #1. (b) Nearest neighbor interpolation on the LR frame #1. (c) Bilinear interpolation on the LR frame #1. (d) Average of the bilinear interpolation of all 60 LR frames. (e) Super resolution result of frame #1 in $d = 2$ dimensions of $n = 32$ pixels in every dimension, zooming factor $z = 2$, total number of LR frames $m = 60$, regularization parameters $\alpha_i=4000$, $i = 1, \dots, 60$, $\beta = 10$, $\epsilon = 0.1$, and time steps $\tau_1 = 0.002$, $\tau_2 = 0.02$, where 30 iterations are applied.

6. CONCLUDING REMARKS

Some recent promising advances in the problem of image super-resolution have focussed on combining the super-resolution reconstruction with motion estimation. Accurate motion estimation, however, still represents a fundamental challenge in this problem. Existing approaches have employed only parametric motion models, e.g., affine transformations [20]. In this paper, we have examined the coupled super-resolution problem with a *non-parametric* motion model. The coupled problem is first viewed in the variational sense, involving the minimization of an objective function which is composed of a “consistency” or distance term \mathcal{C} and a regularization expression \mathcal{R} . Diffusion registration was used for the motion estimation and total variation regularization for the super-resolved image. A PDE approach then yields a numerical scheme for the solution of this problem.

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APPENDIX 1: PROOF OF THEOREM 1

Proof. Split the variation $v(x)$ as

$$v(x) := (v_1(x), \dots, v_m(x), g(x))$$

in which $v_i(x) \in \mathbb{R}^d$, $i = 1, \dots, m$ and $g(x) \in \mathbb{R}$. Also define,

$$p_i(x) := \mathcal{H}^*[\mathcal{H}\mathcal{S}_i f(x) - y_i(x)] \nabla \mathcal{S}_i f(x), \quad i = 1, \dots, m$$

$$q(x) := - \sum_{i=1}^m \mathcal{S}_i^* \mathcal{H}^*[\mathcal{H}\mathcal{S}_i f(x) - y_i(x)].$$

Using the Taylor expansion of $f_{u_i+hv_i}(x)$ and $g_{u_i+hv_i}(x)$ with respect to h at the point $x - u(x)$,

$$f_{u_i+hv_i}(x) = f(x - u_i(x) - hv_i(x))$$

$$= f_{u_i}(x) - h \langle \nabla f_{u_i}(x), v_i(x) \rangle_{\mathbb{R}^d} + \mathcal{O}(h^2),$$

$$g_{u_i+hv_i}(x) = g(x - u_i(x) - hv_i(x))$$

$$= g_{u_i}(x) - h \langle \nabla g_{u_i}(x), v_i(x) \rangle_{\mathbb{R}^d} + \mathcal{O}(h^2),$$

for every $i = 1, \dots, m$. Directly using the definitions and the linearity of \mathcal{H}

$$\begin{aligned} & d\mathcal{C}[y; (u, f); v] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} (\mathcal{C}[y; (u, f) + hv] - \mathcal{C}[y; (u, f)]) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} (\mathcal{C}[y; (u_1 + hv_1, \dots, u_m + hv_m, f + hg)] \\ &\quad - \mathcal{C}[y; (u_1, \dots, u_m, f)]) \\ &= \lim_{h \rightarrow 0} \frac{1}{2h} \sum_{i=1}^m \int_{\Omega} [y_i(x) - \mathcal{H}(f + hg)_{u_i+hv_i}(x)]^2 \\ &\quad - [y_i(x) - \mathcal{H}f_{u_i}(x)]^2 dx \\ &= \lim_{h \rightarrow 0} \frac{1}{2h} \sum_{i=1}^m \int_{\Omega} [y_i(x) - \mathcal{H}f_{u_i+hv_i}(x) - h\mathcal{H}g_{u_i+hv_i}(x)]^2 \\ &\quad - [y_i(x) - \mathcal{H}f_{u_i}(x)]^2 dx \\ &= \lim_{h \rightarrow 0} \frac{1}{2h} \sum_{i=1}^m \int_{\Omega} [y_i(x) - \mathcal{H}(f_{u_i}(x) - h \langle \nabla f_{u_i}(x), v_i(x) \rangle_{\mathbb{R}^d} \\ &\quad + \mathcal{O}(h^2)) - h\mathcal{H}g_{u_i}(x) + \mathcal{O}(h^2)]^2 \\ &\quad - [y_i(x) - \mathcal{H}f_{u_i}(x)]^2 dx \\ &= \lim_{h \rightarrow 0} \frac{1}{2h} \sum_{i=1}^m \int_{\Omega} 2[y_i(x) - \mathcal{H}f_{u_i}(x)][h \\ &\quad \mathcal{H}(\langle \nabla f_{u_i}(x), v_i(x) \rangle_{\mathbb{R}^d} - g_{u_i}(x)) + \mathcal{O}(h^2)] dx \\ &= - \sum_{i=1}^m \int_{\Omega} [\mathcal{H}f_{u_i}(x) - y_i(x)] \\ &\quad \mathcal{H}(\langle \nabla f_{u_i}(x), v_i(x) \rangle_{\mathbb{R}^d} - g_{u_i}(x)) dx \\ &= - \sum_{i=1}^m \int_{\Omega} \left[[\mathcal{H}f_{u_i}(x) - y_i(x)] \mathcal{H}(\langle \nabla f_{u_i}(x), v_i(x) \rangle_{\mathbb{R}^d}) \right] dx \\ &\quad + \sum_{i=1}^m \int_{\Omega} [\mathcal{H}f_{u_i}(x) - y_i(x)] \mathcal{H}g_{u_i}(x) dx. \end{aligned}$$

Hence,

$$\begin{aligned} & d\mathcal{C}[y; (u, f); v] \\ &= - \sum_{i=1}^m \left\langle [\mathcal{H}f_{u_i}(x) - y_i(x)], \right. \\ &\quad \left. \mathcal{H}(\langle \nabla f_{u_i}(x), v_i(x) \rangle_{\mathbb{R}^d}) \right\rangle_{\mathcal{L}^2(\Omega)} \\ &\quad + \sum_{i=1}^m \left\langle [\mathcal{H}f_{u_i}(x) - y_i(x)], \mathcal{H}g_{u_i}(x) \right\rangle_{\mathcal{L}^2(\Omega)} \\ &= - \sum_{i=1}^m \left\langle \mathcal{H}^*[\mathcal{H}f_{u_i}(x) - y_i(x)], \right. \\ &\quad \left. (\langle \nabla f_{u_i}(x), v_i(x) \rangle_{\mathbb{R}^d}) \right\rangle_{\mathcal{L}^2(\Omega)} \\ &\quad + \sum_{i=1}^m \left\langle \mathcal{H}^*[\mathcal{H}f_{u_i}(x) - y_i(x)], g_{u_i}(x) \right\rangle_{\mathcal{L}^2(\Omega)} \end{aligned}$$

Therefore,

$$\begin{aligned}
& d\mathcal{C}[y; (u, f); v] \\
= & - \sum_{i=1}^m \left\langle \mathcal{H}^*[\mathcal{H}f_{u_i}(x) - y_i(x)], \right. \\
& \left. \langle (\nabla f_{u_i}(x), v_i(x))_{\mathbb{R}^d} \rangle \right\rangle_{\mathcal{L}^2(\Omega)} \\
& + \sum_{i=1}^m \left\langle \mathcal{H}^*[\mathcal{H}f_{u_i}(x) - y_i(x)], \mathcal{S}_i g(x) \right\rangle_{\mathcal{L}^2(\Omega)} \\
= & - \sum_{i=1}^m \left\langle \mathcal{H}^*[\mathcal{H}f_{u_i}(x) - y_i(x)], \right. \\
& \left. \langle (\nabla f_{u_i}(x), v_i(x))_{\mathbb{R}^d} \rangle \right\rangle_{\mathcal{L}^2(\Omega)} \\
& + \sum_{i=1}^m \left\langle \mathcal{S}_i^* \mathcal{H}^*[\mathcal{H}f_{u_i}(x) - y_i(x)], g(x) \right\rangle_{\mathcal{L}^2(\Omega)} \\
= & - \sum_{i=1}^m \int_{\Omega} \langle p_i(x), v_i(x) \rangle_{\mathbb{R}^d} dx \\
& + \sum_{i=1}^m \int_{\Omega} \mathcal{S}_i^* \mathcal{H}^*[\mathcal{H}f_{u_i}(x) - y_i(x)] g(x) dx \\
= & - \int_{\Omega} \left[\sum_{i=1}^m \langle p_i(x), v_i(x) \rangle_{\mathbb{R}^d} \right. \\
& \left. - \sum_{i=1}^m \mathcal{S}_i^* \mathcal{H}^*[\mathcal{H}f_{u_i}(x) - y_i(x)] g(x) \right] dx \\
= & - \int_{\Omega} \left[\sum_{i=1}^m \langle p_i(x), v_i(x) \rangle_{\mathbb{R}^d} + q(x) g(x) \right] dx \\
= & - \int_{\Omega} \langle p_1(x), v_1(x) \rangle_{\mathbb{R}^d} + \dots \\
& + \langle p_m(x), v_m(x) \rangle_{\mathbb{R}^d} + \langle q(x), g(x) \rangle_{\mathbb{R}} dx \\
= & - \int_{\Omega} \left\langle \left(p_1(x), \dots, p_m(x), q(x) \right), \right. \\
& \left. \left(v_1(x), \dots, v_m(x), g(x) \right) \right\rangle_{\mathbb{R}^{m+1}} dx \\
= & - \int_{\Omega} \langle \Phi(x, u(x), f(x)), v(x) \rangle_{\mathbb{R}^{m+1}} dx.
\end{aligned}$$

□

APPENDIX 2: DEFINITION OF A

$\mathbf{A} \in \mathbb{R}^{n^d \times n^d}$ is defined as $\mathbf{A} := \sum_{l=1}^d \mathbf{A}_l$ where $\mathbf{A}_l = \underbrace{\mathbf{I} \otimes \dots \otimes \mathbf{I}}_{l-1 \text{ times}} \otimes \mathbf{B} \otimes \underbrace{\mathbf{I} \otimes \dots \otimes \mathbf{I}}_{d-l \text{ times}}$, in which $\mathbf{I} \in \mathbb{R}^{n \times n}$ is identity matrix and \otimes denotes the Kronecker product of matrices. The l^{th} factor $\mathbf{B} \in \mathbb{R}^{n \times n}$ is an approximation of the second order derivative in only one spatial direction.

More precisely, it can be defined as the tridiagonal matrix

$$\mathbf{B} = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & -2 & 1 \\ 0 & \dots & \dots & 1 & -2 \end{pmatrix}.$$

APPENDIX 3: CONSTRUCTION OF K, D, AND H

Given the zooming factor $z \in \mathbb{N}$, define $\mathbf{K} \in \mathbb{R}^{(n-z+1)^d \times n^d}$ as

$$\mathbf{K} := \underbrace{\mathbf{K}_1 \otimes \dots \otimes \mathbf{K}_1}_{d \text{ times}}$$

in which $\mathbf{K}_1 \in \mathbb{R}^{(n-z+1) \times n}$ is

$$\mathbf{K}_1 = \frac{1}{z} \begin{pmatrix} \underbrace{1 \dots 1}_{z \text{ times}} \underbrace{0 \dots 0}_{n-z \text{ times}} & 0 \\ 0 \underbrace{1 \dots 1}_{z \text{ times}} \underbrace{0 \dots 0}_{n-z-1 \text{ times}} & 0 \\ 0 \underbrace{0 \underbrace{1 \dots 1}_{z \text{ times}} \underbrace{0 \dots 0}_{n-z-2 \text{ times}}}_{z \text{ times}} & 0 \\ \vdots & \vdots \\ 0 \underbrace{\dots \dots 0}_{n-z \text{ times}} \underbrace{1 \dots 1}_{z \text{ times}} \end{pmatrix}.$$

Also, $\mathbf{D} \in \mathbb{R}^{\lfloor n/z \rfloor^d \times (n-z+1)^d}$ is defined as

$$\mathbf{D} := \underbrace{\mathbf{D}_1 \otimes \dots \otimes \mathbf{D}_1}_{d \text{ times}},$$

where $\mathbf{D}_1 \in \mathbb{R}^{\lfloor n/z \rfloor \times (n-z+1)}$ is

$$\mathbf{D}_1 = \begin{pmatrix} 1 \underbrace{0 \dots 0}_{n-z \text{ times}} & 0 \\ \underbrace{0 \dots 0}_{z \text{ times}} 1 \underbrace{0 \dots 0}_{n-2z \text{ times}} & 0 \\ \underbrace{0 \dots 0}_{2z \text{ times}} & 1 \underbrace{0 \dots 0}_{n-3z} \\ \vdots & \vdots \\ \underbrace{0 \dots 0}_{(\lfloor n/z \rfloor - 1)z \text{ times}} & 1 \underbrace{0 \dots 0}_{n - (\lfloor n/z \rfloor)z \text{ times}} \end{pmatrix}.$$

Finally, $\mathbf{H} \in \mathbb{R}^{\lfloor n/z \rfloor^d \times n^d}$ is defined as the matrix multiplication of $\mathbf{H} := \mathbf{D} \times \mathbf{K}$.